Braid Groups

Three Intertwined Perspectives

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Directed Reading Program, Spring 2023

June 7, 2023

Preliminaries

- A binary structure (G, *) is called a **group** if it satisfies the following axioms:
 - (x * y) * z = x * (y * z), for all $x, y, z \in G$, i.e. * is associative.
 - There exists an identity element denoted by 1_G such that $g * 1_G = g = 1_G * g$.
 - For every element g ∈ G, there exists an element g⁻¹ ∈ G, called the inverse of g, such that g * g⁻¹ = 1_G = g⁻¹ * g.

Example: $(\mathbb{Z}, +)$

Braids

- The braid group on n strands, denoted B_n, is the group of equivalence classes of n-braids.
- We use σ_i to describe the braid in which the *i*-string crosses over the *i*+1-string.
- \blacktriangleright The braid group B_n has the presentation

$$B_n = \langle \sigma_1, \sigma_2, \cdots, \sigma_{n-1} | \sigma_i \sigma_j = \sigma_j \sigma_i \text{ for } |i-j| > 1 \text{ and} \\ \sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1} \text{ for } 1 \le i \le n-2 \rangle.$$

Braids

- The map $\pi: B_n \to S_n$ is a homomorphism.
- An important subgroup of B_n is the **pure braid group** PB_n. It is the set of all braids in B_n such that the strings line up in the same order on the left as they do on the right.
- ln fact, ker $(\pi) = PB_n$.
- We can describe a process called **combing** using a "standard" generating set for PB_n. From this, we obtain the braid's normal form.
- If two pure braids are equivalent, then they have the same combing.

Theorem.

 B_n has solvable word problem.

Configuration Spaces

A configuration space is the set of all possible ordered configurations of n particles

$$C_n(\mathbb{R}^2) = \{(p_1, p_2, \cdots, p_n) \in (\mathbb{R}^2)^n \mid p_i \neq p_j \text{ for } i \neq j\},\$$

where $p_i \neq p_j$ is the condition that the particles must not collide. We can also consider the set of all possible *unordered* configurations of *n* particles

$$UC_n(\mathbb{R}^2) = \{\{p_1, p_2, \cdots, p_n\} \subset \mathbb{R}^2 \mid p_i \neq p_j \text{ for } i \neq j\}$$

▶ The fundamental group of $UC_n(\mathbb{R}^2)$ is isomorphic to the braid group B_n .

Punctured Disks

- Let S be an orientable surface. The mapping class group of S, denoted by MCG(S), is the group of homotopy classes of orientation-preserving homeomorphisms of S.
- ▶ Denote D_n as a disk with *n* punctures. MCG(D_n) is the group of mapping classes of homeomorphisms of an *n*-punctured disk which fix points on the boundary circle pointwise, but not necessarily the *n* punctures.
- $MCG(D_n)$ is isomorphic to the braid group B_n .

Theorem.

 B_n is torsion free.

A Word on the Nielsen-Thurston Classification

Using this mapping class group interpretation of braids, each braid can be classified as *periodic, reducible* or *pseudo-Anosov*.

We love mapping class groups!

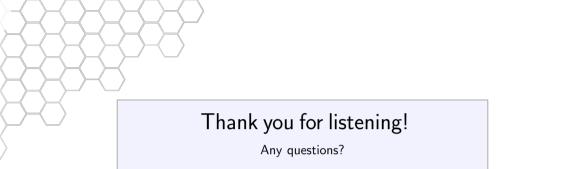
References

M.Clay, D. Margalit (2017)

Office Hours with a Geometric Group Theorist. Princeton University Press, 2017.

B. Farb, D. Margalit (2012)

A Primer on Mapping Class Groups. Princeton University Press, 2012.



Special thanks to Greyson Meyer & DRP Organizers :)