# Braid Groups

Three Intertwined Perspectives

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# Preliminaries

- A binary structure $(G, *)$ is called a **group** if it satisfies the following axioms:
  - $(x * y) * z = x * (y * z)$, for all $x, y, z \in G$, i.e. $*$ is associative.
  - There exists an identity element denoted by $1_G$ such that $g * 1_G = g = 1_G * g$.
  - For every element $g \in G$, there exists an element $g^{-1} \in G$, called the inverse of $g$, such that $g * g^{-1} = 1_G = g^{-1} * g$.
- Example: $(\mathbb{Z}, +)$

# Braids

- The **braid group** on $n$ strands, denoted $B_n$, is the group of equivalence classes of $n$-braids.
- We use $\sigma_i$ to describe the braid in which the $i-$string crosses over the $i+1-$string.
- The braid group $B_n$ has the presentation

$$B_n = \langle \sigma_1, \sigma_2, \cdots, \sigma_{n-1} \,|\, \sigma_i\sigma_j = \sigma_j\sigma_i \text{ for } |i-j| > 1 \text{ and } \\ \sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1} \text{ for } 1 \leq i \leq n-2 \rangle.$$

# Braids

- The map $\pi : B_n \to S_n$ is a homomorphism.
- An important subgroup of $B_n$ is the **pure braid group** $PB_n$. It is the set of all braids in $B_n$ such that the strings line up in the same order on the left as they do on the right.
- In fact, $\ker(\pi) = PB_n$.
- We can describe a process called **combing** using a "standard" generating set for $PB_n$. From this, we obtain the braid's normal form.
- If two pure braids are equivalent, then they have the same combing.

Theorem.

$B_n$ has solvable word problem.

# Configuration Spaces

- A **configuration space** is the set of all possible *ordered* configurations of $n$ particles

$$C_n(\mathbb{R}^2) = \{(p_1, p_2, \cdots, p_n) \in (\mathbb{R}^2)^n \mid p_i \neq p_j \text{ for } i \neq j\},$$

  where $p_i \neq p_j$ is the condition that the particles must not collide. We can also consider the set of all possible *unordered* configurations of $n$ particles

$$UC_n(\mathbb{R}^2) = \{\{p_1, p_2, \cdots, p_n\} \subset \mathbb{R}^2 \mid p_i \neq p_j \text{ for } i \neq j\}.$$

- The fundamental group of $UC_n(\mathbb{R}^2)$ is isomorphic to the braid group $B_n$.

# Punctured Disks

- Let $S$ be an orientable surface. The **mapping class group** of $S$, denoted by $\mathrm{MCG}(S)$, is the group of homotopy classes of orientation-preserving homeomorphisms of $S$.
- Denote $D_n$ as a disk with $n$ punctures. $\mathrm{MCG}(D_n)$ is the group of mapping classes of homeomorphisms of an $n-$punctured disk which fix points on the boundary circle pointwise, but not necessarily the $n$ punctures.
- $\mathrm{MCG}(D_n)$ is isomorphic to the braid group $B_n$.

**Theorem.**
$B_n$ is torsion free.

# A Word on the Nielsen-Thurston Classification

Using this mapping class group interpretation of braids, each braid can be classified as *periodic*, *reducible* or *pseudo-Anosov*.

We love mapping class groups!

# References


M.Clay, D. Margalit (2017)
Office Hours with a Geometric Group Theorist. Princeton University Press, 2017.

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A Primer on Mapping Class Groups. Princeton University Press, 2012.

# Thank you for listening!

Any questions?

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